# On the Use of Differentials in the Investigation of the real Roots of Equations *

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**§294** The properties of maxima and minima provide us with a new method to find the nature of roots of equations, i.e., whether they are real or imaginary. For, let an equation of any order be propounded

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0,$$

whose roots we put to be $p$, $q$, $r$, $s$, $t$ etc. such that $p$ is the smallest, $q$ the second smallest and likewise for the remaining roots, i.e. they are already ordered according to their magnitude; of course, $q > p$, $r > q$, $s > r$, $t > s$ etc. But let us assume that all roots of the equations are real and the largest exponent $n$ will at the same time be the number of roots $p$, $q$, $r$ etc. Additionally, let us assume all these roots to be different; this does not exclude the case of equal roots, since non equal roots become equal, if their difference is assumed to be infinitely small.

**§295** Since the propounded expression $x^n - Ax^{n-1} + \text{etc.}$ is zero only, if any of the values $p$, $q$, $r$ etc. is substituted for $x$, but does not vanish in all remaining cases, let us put in general

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$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \text{etc.} = z,$$

such that $z$ can be considered as a function of $x$. Now let us assume that determined values are substituted successively for $x$ starting from the smallest $x = -\infty$ and then going over to continuously larger values; it is perspicuous that $z$ will hence have values either greater than zero or smaller than zero and will just vanish, if one puts $x = p$; in this case it will be $z = 0$. Increase the values of $x$ beyond $p$ and the values of $z$ will either become positive or negative until one gets to the value $x = q$; in this case it will again be $z = 0$. Therefore, it is necessary that, because the values of $z$ from 0 get back to 0 again, $z$ had a maximum or minimum value within these limits, a maximum of course, if the values of $z$ were positive, while $x$ lies within the limits $p$ and $q$, a minimum, if they were negative. In like manner, while $x$ is increased, and becomes larger than $q$, up to $r$, the function $z$ will take on a maximum or a minimum value, a maximum value, of course, if it was a minimum value before, and vice versa. For, above [§ 263] we saw that the maxima and minima alternate.

**§296** Hence, because within the limits determined by each two roots of $x$ there is a value, for which the function $z$ becomes a minimum or a maximum, the number of maxima and minimum values the function $z$ has, will be smaller than the number of real roots by 1; and they will alternate that the maximal values of $z$ are positive and the minimal values are negative. If vice versa the function $z$ has a maximum or at least a positive value in the case $x = f$ and a minimum value or at least a negative value in the case $x = g$, since, if the values of $x$ go over from $f$ into $g$, the function $z$ goes over from the positive into the negative values, it is necessary that in between it passes through 0, and therefore a root of $x$ will be contained within the limits $f$ and $g$. But if this condition is not satisfied that the maximum and minimum values of $z$ alternately become positive and negative, that conclusion is incorrect. For, if the function $z$ has minima which are also positive, it can happen that the value of $z$ goes over from a maximum into the following minimum, although is does not vanish in between. Furthermore, from the things we explained it is understood, even though not all roots of the propounded equation were real, that nevertheless within the limits determined by each two roots there is always a maximum or a minimum, even though the converse proposition does not hold in general that there is always a real root within the limits determined by each two maxima or minima; but it holds, if the following condition is added: If the one value of $z$ was positive, the other is negative.

**§297** Therefore, since we saw above that the values of $x$, for which the function

$$z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.}$$

has a maximum or a minimum value, are the roots of this differential equation

$$\frac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.}$$

it is obvious, if all roots of the equation $z = 0$, whose number is $= n$, were real, that also all roots of the equation $\frac{dz}{dx} = 0$ will be real. For, because the function $z$ has as many maxima or minima, as the number $n-1$ contains units, the equation $\frac{dz}{dx} = 0$ necessarily has the same amount of roots; and hence all its roots will be real. From this it is at the same time understood that the function $z$ cannot have more than $n-1$ maxima or minima. Therefore, we have this very far extending rule: If all roots of the equation $z = 0$ were real, then also the equation $\frac{dz}{dx} = 0$ will have only real roots. Hence it vice versa follows, if not all roots of the equation $\frac{dz}{dx} = 0$ were real, that then also not all roots of the equation $z = 0$ will be real.

**§298** Since within the limits determined by each two real roots of the equation $z = 0$ there is always one value for which the function $z$ has a maximum or a minimum value, it follows, if the equation $z = 0$ has two real roots, that the equation $\frac{dz}{dx} = 0$ will necessarily have one real root. In like manner, if the equation $z = 0$ has three real roots, then the equation $\frac{dz}{dx} = 0$ will certainly have two real roots. And in general, if the equation $z = 0$ has $m$ real roots, it is necessary that at least $m-1$ roots of the equation $\frac{dz}{dx} = 0$ are real. Hence, if the equation $\frac{dz}{dx} = 0$ has less than $m-1$ real roots, then vice versa the equation $z = 0$ has less than $m$ real roots. But the converse is not true; for, even though the differential equation $\frac{dz}{dx} = 0$ has several or even only real roots, it does nevertheless not follow that the equation $z = 0$ will have even one real root. For, it can happen that all the roots of the equation $\frac{dz}{dx} = 0$ real, although all roots of the equation $z = 0$ are imaginary.

**§299** Nevertheless, if the condition mentioned above is added, the converse proposition can be stated in such a way that from the real roots of the equation $\frac{dz}{dx} = 0$ the number of real roots of the equation $z = 0$ is known for certain. For, let us put that $\alpha$, $\beta$, $\gamma$, $\delta$ etc. are real roots of the equation $\frac{dz}{dx} = 0$, of which $\alpha$ is the largest; but the remaining follow in the order of their magnitude. Therefore, having substituted these values for $x$, the function $z$ will have either maximum or minimum values alternately. But because the function $z$ becomes $= \infty$, if one puts $x = \infty$, it is plain that its values have to decrease continuously, while the values of $x$ are decreased from $\infty$ to $\alpha$; therefore, $z$ will have a minimum value in the case $x = \alpha$. Therefore, if in this case $x = \alpha$ the function $z$ has a negative value, it is necessary that it was $= 0$ somewhere else before, and so a real root $x > \alpha$ of the equation $z = 0$ must exist; but if for $x = \alpha$ the function $z$ still has a positive value, it can not be smaller before; for, otherwise also a minimum would exist, before $x$ was decreased to $\alpha$, contradicting the hypothesis; hence the equation $z = 0$ can have no real root larger than $\alpha$. Therefore, if we put that for $x = \alpha$ we have $z = \mathfrak{A}$, one can decide this way: If $\mathfrak{A}$ was a positive quantity, the equation $z = 0$ will have no real root larger than $\alpha$; but if $\mathfrak{A}$ was a negative quantity, the equation $z = 0$ will always have one real root larger than $\alpha$ but not more.

**§300** To make a further decision

| if it is put | let |
|---|---|
| $x = \alpha$ | $z = \mathfrak{A}$ |
| $x = \beta$ | $z = \mathfrak{B}$ |
| $x = \gamma$ | $z = \mathfrak{C}$ |
| $x = \delta$ | $z = \mathfrak{D}$ |
| $x = \varepsilon$ | $z = \mathfrak{E}$ |
| etc. | etc. |

Therefore, because $\mathfrak{A}$ was a minimum, $\mathfrak{B}$ will be a maximum, and if $\mathfrak{A}$ was positive, also $\mathfrak{B}$ will be positive and therefore no real root of the equation $z = 0$ will exist within the limits $\alpha$ and $\beta$ . Hence, if this equation has no real root greater than $\alpha$, it will also not have one, which is greater than $\beta$. But if $\mathfrak{A}$ was a negative, in which case one root $x > \alpha$ of the equation is given, see, whether the value of $\mathfrak{B}$ is positive or negative. In the first case a there will be a root $x > \beta$, in the second there will be no root contained within the limits $\alpha$ and $\beta$. In like manner, because $\mathfrak{B}$ was a maximum, $\mathfrak{C}$ will be a minimum; hence, if $\mathfrak{B}$ had a negative value, $\mathfrak{C}$ will be a much more negative and in this case there will be no root contained within the limits $\beta$ and $\gamma$. But if $\mathfrak{B}$ was positive, there will be a real root within the limits $\beta$ and $\gamma$, if $\mathfrak{C}$ becomes negative; but if $\mathfrak{C}$ is also positive, then there will be no root contained within the limits $\beta$ and $\gamma$ and in like manner the decision is to be made for the other cases.

**§301** That these criteria for the decision seen more clearly, I summarized them in the following table.

| The equation $z = 0$ will have one single real root | | | | | |
|---|---|---|---|---|---|
| which is contained within the limits | | | if it was | | |
| $x = \infty$ | and | $x = \alpha$ | $\mathfrak{A} = -$ | | |
| $x = \alpha$ | and | $x = \beta$ | $\mathfrak{A} = -$ | and | $\mathfrak{B} = +$ |
| $x = \beta$ | and | $x = \gamma$ | $\mathfrak{B} = +$ | and | $\mathfrak{C} = -$ |
| $x = \gamma$ | and | $x = \delta$ | $\mathfrak{C} = -$ | and | $\mathfrak{D} = +$ |
| $x = \delta$ | and | $x = \varepsilon$ | $\mathfrak{D} = +$ | and | $\mathfrak{E} = -$ |
| | etc. | | | etc. | |

The negated converses of these propositions will hold in like manner.

| The equation $z = 0$ will have no real root which is contained within the limits | | | if it was not | | |
|---|---|---|---|---|---|
| $x = \infty$ | and | $x = \alpha$ | $\mathfrak{A} = -$ | | |
| $x = \alpha$ | and | $x = \beta$ | $\mathfrak{A} = -$ | and | $\mathfrak{B} = +$ |
| $x = \beta$ | and | $x = \gamma$ | $\mathfrak{B} = +$ | and | $\mathfrak{C} = -$ |
| $x = \gamma$ | and | $x = \delta$ | $\mathfrak{C} = -$ | and | $\mathfrak{D} = +$ |
| $x = \delta$ | and | $x = \varepsilon$ | $\mathfrak{D} = +$ | and | $\mathfrak{E} = -$ |
| | etc. | | | etc. | |

Therefore, by means of these rules, from the roots of the equation $\frac{dz}{dx} = 0$, if they were known, not only the number of real roots of the equation $z = 0$ is concluded, but also the limits will become known within which each root is contained.

## EXAMPLE

*Let this equation be propounded* $x^4 - 14xx + 24x - 12 = 0$; *it is in question, whether it has real roots and how many.*

The differential equation will be $4x^3 - 28x + 24 = 0$ or $x^3 - 7x + 6 = 0$, whose roots are 1, 2 and $-3$, which, ordered according to their magnitude, will give

| | whence it will be |
|---|---|
| $\alpha = +2$ | $\mathfrak{A} = -4$ |
| $\beta = +1$ | $\mathfrak{B} = -1$ |
| $\gamma = -3$ | $\mathfrak{C} = -129.$ |

Because of the negative $\mathfrak{A}$, the propounded equation will have a real root $> 2$, but, because of the negative $\mathfrak{B}$, it will neither have a real root within the limits 2 and 1 nor within the limits 1 and $-3$. But because for $x = -3$, $z = \mathfrak{C} = -129$, and if one sets $x = -\infty$, $z = +\infty$, it is necessary that there is a real root within the limits $-3$ and $-\infty$. Therefore, the propounded equation will have two real roots, the one $x > 2$, the other $x < -3$; hence two roots will be imaginary. Therefore, it has to be decided from the last maximum or minimum of the propounded equation as from the first. If the propounded equation was of even order, the last maximum or minimum (it will be a minimum in this case), if it was negative, indicates a real root, if positive, an imaginary root. But for the equations of odd degree, since for $x = -\infty$ also $z = -\infty$, if the last maximum was positive, a real root is indicated, if negative, an imaginary one.

**§302** Therefore, the rule for discovering the real and imaginary roots can be expressed conveniently this way. Having propounded any equation $z = 0$, consider its differential $\frac{dz}{dx} = 0$, whose real roots, ordered according to their magnitude, we want to be $\alpha$, $\beta$, $\gamma$, $\delta$ etc.; then, having put

$$x = \alpha, \quad \beta, \quad \gamma, \quad \delta, \quad \varepsilon, \quad \zeta \quad \text{etc.}$$

let

$$z = \mathfrak{A}, \quad \mathfrak{B}, \quad \mathfrak{C}, \quad \mathfrak{D}, \quad \mathfrak{E}, \quad \mathfrak{F}, \quad \text{etc.}$$

Now, if the signs are

$$- \quad + \quad - \quad + \quad - \quad + \quad \text{etc.},$$

the equation $z = 0$ will have as many real roots as one has letters $\alpha$, $\beta$, $\gamma$ etc. and additionally one more. But if one of these capital letters does not have the mentioned sign written under it, two imaginary roots will be indicated. So, if $\mathfrak{A}$ has the sign $+$, there would be no root contained within the limits $\infty$ and $\beta$. If $\mathfrak{B}$ has the sign $-$, no root will lie within the limits $\alpha$ and $\gamma$, and if $\mathfrak{C}$ has the sign $+$, there will be no root within the limits $\beta$ and $\delta$ and so forth. But in general except for the imaginary roots indicated this way the equation $z = 0$ will additionally have as many imaginary roots as the equation $\frac{dz}{dx} = 0$.

**§303** If it happens that one of the values $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ vanishes, then for it the equation $z = 0$ will have two equal roots. If $\mathfrak{A} = 0$, then it will have two roots equal to $\alpha$; if $\mathfrak{B} = 0$, two roots will be equal $= \beta$. For, in this case the equation $z = 0$ will have a root in common with the differential equation $\frac{dz}{dx} = 0$; but above [§ 245] we demonstrated that this is a sign for two equal roots. But if the equation $\frac{dz}{dx} = 0$ has two or more equal roots, then, if their number was even, neither a maximum nor a minimum will be indicated; therefore, for the present task, an even number of equal roots can be neglected. But if the number of equal roots of the equation $\frac{dz}{dx}=0$ was odd, all except for one are to be rejected in the decision, if not by accident in this case also the function $z$ itself vanishes. For, if this happens, the equation $z=0$ will also have equal roots and one more than the equation $\frac{dz}{dx}=0$. So, if it was $\frac{dz}{dx}=(x-\zeta)^nR$ such that this equation has $n$ roots equal to $\zeta$, if $z$ vanishes for $x=\zeta$, the equation $z=0$ will have $n+1$ roots equal to $\zeta$.

**§304** Let us apply these prescriptions to simpler equations and let us begin with the quadratic ones. Therefore, let this equation be propounded

$$z=x^2-Ax+B=0;$$

its differential will be

$$\frac{dz}{dx}=2x-A,$$

having put which $=0$ it will be

$$x=\frac{1}{2}A \quad \text{or} \quad \alpha=\frac{1}{2}A.$$

Substitute this value for $x$ and it will be

$$z=-\frac{1}{4}AA+B=\mathfrak{A};$$

hence, we conclude, if this value of $\mathfrak{A}$ was negative, i.e. if $AA>4B$, that the equation $xx-Ax+B=0$ will have two real roots, the one greater than $\frac{1}{2}A$, the other smaller. But if the value of $\mathfrak{A}$ was positive or $AA<4B$, then both roots of the propounded equation will be imaginary. But if it was $\mathfrak{A}=0$ or $AA=4B$, the propounded equation will have two equal roots, both of them $=\frac{1}{2}A$. Since these things are very well-known from the nature of quadratic equations, the validity of these principles is well-illustrated by this case and at the same time their utility in this task is understood.

**§305** Let us proceed to the investigation of cubic equations in the same way. Therefore, let this equation be propounded

$$x^3-Ax^2+Bx-C=z=0;$$

because its differential quotient is

$$3xx - 2Ax + B = \frac{dz}{dx},$$

if one puts this $= 0$, it will be

$$xx = \frac{2Ax - B}{3},$$

the two roots of which equation are either both imaginary or both equal or both real and not equal. Therefore, because hence

$$x = \frac{A \pm \sqrt{A^2 - 3B}}{3},$$

the two roots will be imaginary, if it was $AA < 3B$; in this case the propounded cubic equation will have one single real root, which lies within limits $+\infty$ and $-\infty$, of course. Now let the two roots be equal to each other or let $AA = 3B$; it will be $x = \frac{A}{3}$. Therefore, if not $z = 0$ at the same time, these two roots are to be considered as one and the equation will have one single real root as before; but if in the case $x = \frac{A}{3}$ $z = 0$ at the same time, what happens, if it was $-\frac{2}{27}A^3 + \frac{1}{3}AB - C = 0$ or $C = \frac{1}{3}AB - \frac{2}{27}A^3$, this means, if it was $B = \frac{1}{3}A^2$ and $C = \frac{1}{27}A^3$, the equation will have three equal roots, each of them $= \frac{1}{3}A$. Now let us expand the third case, in which both roots of the differential equation are real and different from each other, what happens, if $AA > 3B$. Therefore, let $AA = 3B + ff$ or $B = \frac{1}{3}AA - \frac{1}{3}ff$; those two roots will be

$$x = \frac{A \pm f}{3}.$$

Therefore, it will be $\alpha = \frac{1}{3}A + \frac{1}{3}f$ and $\beta = \frac{1}{3}A - \frac{1}{3}f$. Therefore, find the values of $z$ corresponding to these, i.e. $\mathfrak{A}$ and $\mathfrak{B}$, and because both roots are contained in this equation $xx = \frac{2}{3}Ax - \frac{1}{3}B$, it will be

$$z = -\frac{1}{3}Axx + \frac{2}{3}Bx - C = -\frac{2}{9}AAx + \frac{1}{9}AB + \frac{2}{3}Bx - C.$$

Therefore, these equations result

$$\mathfrak{A} = -\frac{2}{27}A^3 + \frac{1}{3}AB - \frac{2}{27}A^2 f + \frac{2}{9}Bf - C = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 - C,$$

$$\mathfrak{B} = -\frac{2}{27}A^3 + \frac{1}{3}AB + \frac{2}{27}A^2 f - \frac{2}{9}Bf - C = \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3 - C$$

because of $B = \frac{1}{3}AA - \frac{1}{3}ff$. Therefore, if $\mathfrak{A}$ was a negative quantity, what happens, if it was $C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$, the equation $z = 0$ will have one single real root $> \alpha$, i.e. larger than $\frac{1}{3}A + \frac{1}{3}f$. Therefore, let us put that

$$C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 \quad \text{or that} \quad C = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 + gg$$

and, as we saw, the propounded cubic equation will have a real root $> \frac{1}{3}A + \frac{1}{3}f$. But of what nature the remaining roots are, will be understood from the value $\mathfrak{B}$; but it will be $\mathfrak{B} = \frac{4}{27}f^3 - gg$; if it was positive, the equation will have two additional real roots, the first contained within the limits $\alpha$ and $\beta$, i.e. within $\frac{1}{3}A + \frac{1}{3}f$ and $\frac{1}{3}A - \frac{1}{3}f$, but the one smaller than $\frac{1}{3}A - \frac{1}{3}f$. But if it was $gg > \frac{4}{27}f^3$ or $\mathfrak{B}$ was negative, the equation will have two imaginary roots. But if it was $\mathfrak{B} = 0$ or $\frac{4}{27}f^3 = gg$, the two roots will become equal, both of them $= \beta = \frac{1}{3}A - \frac{1}{3}f$. Finally, if the value of $\mathfrak{A}$ is positive or $C < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$, the equations will have two imaginary roots and the third will be real and $> \frac{1}{3}A - \frac{1}{3}f$. And if the value of $\mathfrak{A}$ is $= 0$, two roots will be equal and $= \alpha$, while the third remains $< \frac{1}{3}A - \frac{1}{3}f$.

**§306** Therefore, for all three roots of the cubic equation $x^3 - Ax^2 + Bx - C = 0$ to be real, three conditions are required. First, that

$$B < \frac{1}{3}AA;$$

therefore, $B = \frac{1}{3}AA - \frac{1}{3}ff$. Secondly, that

$$C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3.$$

Thirdly, that

$$C < \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3.$$

These two last inequalities reduce to the single one that $C$ is contained within the limits

$$\frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 \quad \text{and} \quad \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3$$

or within these limits

$$\frac{1}{27}(A+f)(A-2f) \quad \text{and} \quad \frac{1}{27}(A-f)^2(A+2f).$$

Therefore, if one of these conditions is missing, the equation will have two imaginary roots. So, if it was $A=3$, $B=2$, it will be $\frac{1}{3}ff = \frac{1}{3}AA - B = 1$ and $ff = 3$; therefore, this equation $x^3 - 3xx + 2x - C = 0$ can have only real roots, if $C$ is contained within the limits $-\frac{2\sqrt{3}}{9}$ and $+\frac{2\sqrt{3}}{9}$. Hence, if it was either $C < -\frac{2\sqrt{3}}{9}$ or $C < -0.3849$ or $C > +\frac{2\sqrt{3}}{9}$ or $C > 0.3849$ or together $CC > \frac{4}{27}$, the equation will have one single real root.

**§307** Since in each equation the second term can be thrown out, let us put that $A=0$ such that we have this cubic equation

$$x^3 + Bx - C = 0.$$

Therefore, for all three roots of this equation to be real, first it is necessary that $B<0$ or $B$ must be a negative quantity. Therefore, let $B=-kk$; it will be $ff = 3kk$ and additionally it is required that the quantity $C$ is contained within the limits $-\frac{2}{27}f^3$ and $+\frac{2}{27}f^3$, this means within $-\frac{2}{9}kk\sqrt{3kk}$ and $+\frac{2}{9}kk\sqrt{3kk}$. Therefore, it will be $CC < \frac{4}{27}k^6$ or $CC < -\frac{4}{27}B^3$. Therefore, the nature of cubic equations, which have three real roots, can be expressed as one single condition, if we say that

$$4B^3 + 27CC$$

is a negative quantity. For, it is necessary that $B$ is a negative quantity, since otherwise $4B^3 + 27CC$ can not become negative. Therefore, in general we can affirm that the equation $x^3 + Bx \pm C = 0$ only has real roots, if $4B^3 + 27CC$ was a negative quantity; but if this quantity was positive, one will be real, the other two imaginary; but if $4B^3 + 27CC = 0$, all roots will be real, but two will be equal to each other.

**§308** Let us proceed to fourth order equations, in which we want to assume the second term to be missing. Therefore, let

$$x^4 + Bx^2 - Cx + D = 0.$$

let us set $x = \frac{1}{u}$ and it will be

$$1 + Bu^2 - Cu^3 + Du^4 = 0,$$

the differential of which equation is

$$2Bu - 3Cu^2 + 4Du^3 = 0,$$

which has the one single root $u = 0$; but then it will be

$$uu = \frac{6Cu - 4B}{8B}$$

and

$$u = \frac{3C \pm \sqrt{9CC - 32BD}}{8D}.$$

Therefore, for all four roots to be real, first it is required that $9CC > 32BD$. Therefore, let us put that $9CC = 32BD + 9ff$; it will be $u = \frac{3C \pm 3f}{8D}$. Here, $C$ can always be assumed to be a positive quantity; for, if it was not such a one, putting $u = -v$ it will become one. But soon we will demonstrate that not all roots can be real, if $B$ is not a negative quantity. Therefore, let $B = -gg$ and it will be

$$9CC = 9ff - 32ggD \quad \text{and} \quad u = \frac{3C \pm 3f}{8D}.$$

Two cases are to be considered, depending on whether $D$ is a positive or negative quantity.

I. Let $D$ be a positive quantity and it will be $f > C$ and the three roots of $u$, ordered according to their magnitude, will be

$$1.\ \ u = \frac{3C + 3f}{8D}, \quad 2.\ \ u = 0, \quad 3.\ \ u = \frac{3C - 3f}{8D}.$$

But the equation

$$u^4 - \frac{Cu^3}{D} + \frac{Bu^2}{D} + \frac{1}{D} = 0,$$

having substituted these values for $u$, will give the following three values

$$\mathfrak{A} = \frac{27(C+f)^3(C-3f)}{4096D^4} + \frac{1}{D}, \quad \mathfrak{B} = \frac{1}{D}, \quad \mathfrak{C} = \frac{27(C-f)^3(C+3f)}{4096} + \frac{1}{D},$$

of which the first and the third must be negative; both of them, because of the positive $C$ and $C < f$, become smaller than $\frac{1}{D}$. Therefore, it must be

$$\frac{1}{D} < \frac{27(C+f)^3(3f-C)}{4096D^4} \quad \text{and} \quad \frac{1}{D} < \frac{27(f-C)^3(C+3f)}{4096D^4}$$

or

$$4096D^3 < 27(f+C)^3(3f-C) \quad \text{and} \quad 4096D^3 < 27(f-C)^3(C+3f).$$

But the first quantity is always a lot larger than the second; hence it suffices, if it was $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$, while $B = \frac{9CC-9ff}{32D}$ and $f > C$ and $D > 0$. Therefore, if $D$ was a positive quantity, $C$ positive, $B$ negative, that $f > C$, and $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$, i.e. $D < \frac{3}{16}(f-C)\sqrt[3]{3f+C}$, the equation will have only real roots. But if it was $D > \frac{3}{16}(f-C)\sqrt[3]{3f+C}$, but nevertheless $D < \frac{3}{16}(f+C)\sqrt[3]{3f-C}$, two roots will be real and two imaginary. But if it was $D > \frac{3}{16}(f+C)\sqrt[3]{3f-C}$, all four roots will be imaginary.

Let $D$ be a negative quantity, say $= -F$, while $C$ remains positive and $B$ negative; because of $B = \frac{9CC-9ff}{32D} = \frac{9ff-9CC}{32F}$, it will be $C > f$. Therefore, because $u = \frac{3C\pm 3f}{8D} = -\frac{3C\pm 3f}{8F}$, the three values, ordered according to their magnitude, will be

$$1.\ u = 0, \quad 2.\ u = -\frac{3C-3f}{8F}, \quad 3.\ u = -\frac{3C+3f}{8F},$$

which will give the following values

$$\mathfrak{A} = -\frac{1}{F}, \quad \mathfrak{B} = \frac{27(C-f)^3(C+3f)}{4096F^4} - \frac{1}{F}, \quad \mathfrak{C} = \frac{27(C+f)^3(C-3f)}{4096F^4} - \frac{1}{F}.$$

Therefore, since $\mathfrak{A}$ is a negative quantity, the equation will now certainly have one and therefore also two real roots. But for all roots to be real, it is necessary that $\mathfrak{B}$ is a positive quantity and therefore $27(C-f)^3(C+3f) > 4096F^3$; but then it is necessary that $\mathfrak{C}$ is a negative quantity or $27(C+f)^3(C-3f) < 4096F^3$. Therefore, for all roots to become real, it is required that $F^3$ is contained within the limits

$$\frac{27}{4096}(C+f)^3(C-3f) \quad \text{and} \quad \frac{27}{4096}(C-f)^3(C+3f)$$

or that $F$ is contained within the limits

$$\frac{3}{16}(C+f)\sqrt[3]{C-3f} \quad \text{and} \quad \frac{3}{16}(C-f)\sqrt[3]{C+3f};$$

And if $F$ is not contained within these limits, two roots will be imaginary.

III. Now let us put that $B$ is a positive quantity and $D$ is positive as well; because of $B = \frac{9CC-9ff}{32D}$, it will be $C > f$, and because $u = \frac{3C \pm 3f}{8D}$, the roots, ordered according to their magnitude, will be

$$1.\ u = \frac{3(C+f)}{8D}, \quad 2.\ u = \frac{3(C-f)}{8D} \quad \text{and} \quad u = 0,$$

whence the following values result

$$\mathfrak{A} = \frac{27(C+f)^3(C-3f)}{4096D^4} + \frac{1}{D}, \quad \mathfrak{B} = \frac{27(C-f)^3(C+3f)}{4096D^4} + \frac{1}{D}, \quad \mathfrak{C} = \frac{1}{D};$$

since here $\mathfrak{C}$ is a positive quantity, two roots will certainly be imaginary. But if $\mathfrak{A}$ was negative, what happens, if $4096D^3 < 27(C+f)^3(3f-C)$, two roots will be real; but if it was $4096D^3 > 27(C+f)^3(3f-C)$, then all four roots will be imaginary.

IV. Let $B$ remain positive, but let $D$ be negative $= -F$; because of $B = \frac{9ff-9CC}{32F}$, it will be $f > C$ and, because of $u = -\frac{3C \pm 3f}{8F}$, the three roots of $u$, ordered according to their magnitude, will be

$$1.\ u = \frac{3(f-C)}{8F}, \quad 2.\ u = 0 \quad \text{and} \quad 3.\ u = -\frac{3(C+f)}{8F},$$

whence these values result

$$\mathfrak{A} = -\frac{27(f-C)^3(C+3f)}{4096F} - \frac{1}{F}, \quad \mathfrak{B} = -\frac{1}{F}, \quad \mathfrak{C} = -\frac{27(C+f)^3(3f-C)}{4096F^4} - \frac{1}{F},$$

where, because of the negative $\mathfrak{A}$ and $\mathfrak{C}$, the equation certainly has two real roots, but because of the negative $\mathfrak{B}$ two roots will be imaginary.

**§309** Therefore, if we put the letters $B$, $C$, $D$ to denote positive quantities, the following cases to be distinguished result, which, because of $f = \sqrt{CC - \frac{32}{9}BD}$, reduce to this.

I. If the equation is $x^4 - Bx^2 \pm Cx + D = 0$, all roots will be real, if it was

$$D > \frac{3}{16}\left(\sqrt{CC + \frac{32}{9}BD} - C\right)\sqrt[3]{3\sqrt{CC + \frac{32}{9}BD} + C};$$

two roots will be real and two imaginary, if it was

$$D > \frac{3}{16}\left(\sqrt{CC + \frac{32}{9}BD} - C\right)\sqrt[3]{3\sqrt{CC + \frac{32}{9}BD} + C},$$

but

$$D < \frac{3}{16}\left(\sqrt{CC + \frac{32}{9}BD} + C\right)\sqrt[3]{3\sqrt{CC + \frac{32}{9}BD} - C};$$

but all roots will be imaginary, if it was

$$D > \frac{3}{16}\left(\sqrt{CC + \frac{32}{9}BD} + C\right)\sqrt[3]{3\sqrt{CC + \frac{32}{9}BD} - C}.$$

II. If the equation is $x^4 - Bx^2 \pm Cx - D = 0$, two roots are always real; and the two remaining ones will also be real, if the quantity $D$ is contained within these limits

$$D > \frac{3}{16}\left(\sqrt{CC - \frac{32}{9}BD} + C\right)\sqrt[3]{C - 3\sqrt{CC + \frac{32}{9}BD}}$$

$$D < \frac{3}{16}\left(C - \sqrt{CC - \frac{32}{9}BD}\right)\sqrt[3]{C + 3\sqrt{CC - \frac{32}{9}BD}};$$

But if $D$ is not contained within this limits, the two remaining roots will be imaginary.

III. If the equation is $x^4 + Bx^2 \pm Cx + D = 0$, two roots will always be imaginary; the remaining two will be real, if it was

$$D < \frac{3}{16}\left(\sqrt{CC - \frac{32}{9}BD} + C\right)\sqrt[3]{3\sqrt{CC - \frac{32}{9}BD} - C};$$

but the two remaining ones will also be imaginary, if it was

$$D > \frac{3}{16}\left(\sqrt{CC - \frac{32}{9}BD} + C\right)\sqrt[3]{3\sqrt{CC - \frac{32}{9}BD} - C}.$$

IV. If the equation is $x^4 + Bx^2 \pm Cx - D = 0$, the two roots of this equation will always be real, the two remaining ones on the other hand will always be imaginary.

## EXAMPLE 1

*If this equation is propounded $x^4 - 2xx + 3x + 4 = 0$, let the nature of the roots be in question, i.e. let it be in question, whether they are real or imaginary.*

Since this example extends to the first case, $B = 2$, $C = 3$ and $D = 4$; hence

$$CC + \frac{32}{9}BD = 9 + \frac{32 \cdot 8}{9} = \frac{337}{9} \quad \text{and} \quad \sqrt{CC + \frac{32}{9}BD} = \frac{\sqrt{337}}{3},$$

whence the conditions for all roots to be real are

$$4 < \frac{3}{16}\left(3 + \frac{\sqrt{337}}{3}\right)\sqrt[3]{\sqrt{337} - 3} = \frac{1}{16}(9 + \sqrt{337})\sqrt[3]{\sqrt{337} - 3},$$

$$4 < \frac{3}{16}\left(\frac{\sqrt{337}}{3} - 3\right)\sqrt[3]{\sqrt{337} + 3} = \frac{1}{16}(\sqrt{337} - 3)\sqrt[3]{\sqrt{337} + 3}.$$

Using approximations it has therefore to be examined, whether $4 < \frac{69}{16}$ and $4 < \frac{24}{16}$; hence, because only the first condition holds, the equation will have two real and two imaginary roots.

## EXAMPLE 2

*Let this equation be propounded $x^4 - 9xx + 12x - 4 = 0$.*

Since this example extends to the second case, it will at least have two real roots. To investigate the nature of the remaining ones note that, because of $B = 9$, $C = 12$ and $D = 4$, it will be

$$\sqrt{CC - \frac{32}{9}BD} = \sqrt{144 - 32 \cdot 4} = 4.$$

And therefore one has to check, whether

$$4 > \frac{3}{16} \cdot 16\sqrt[3]{0}, \quad \text{i.e} \quad 4 > 0,$$

and

$$4 < \frac{3}{16} \cdot 8\sqrt[3]{24}, \quad \text{i.e.} \quad 4 < 3\sqrt[3]{3};$$

because both is true, the propounded equation will have four real roots.

## Example 3

*Let this equation be propounded* $x^4 + xx - 2x + 6 = 0.$

Since this equation extends to the third case, two roots will certainly be imaginary. But $B = 1$, $C = 2$ and $D = 6$ and hence

$$\sqrt{CC - \frac{32}{9}BD} = \sqrt{4 - \frac{64}{3}};$$

Since this quantity is imaginary, also the two remaining ones will certainly be imaginary.

## Example 4

*Let this equation be propounded* $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0.$

At first eliminate the second term; substituting $x = y + 1$ it will be

$$\begin{array}{rrcrrrrrrrrr}
 & x^4 & = & y^4 & + & 4y^3 & + & 6yy & + & 4y & + & 1 \\
- & 4x^3 & = & & - & 4y^3 & - & 12y^2 & - & 12y & - & 4 \\
+ & 8x^2 & = & & & & + & 8y^2 & + & 16y & + & 8 \\
- & 16x & = & & & & & & - & 16y & - & 16 \\
+ & 20 & = & & & & & & & & + & 20 \\
\hline
\text{Therefore} & & & y^4 & + & 2yy & - & 8y & + & 9 & = & 0;
\end{array}$$

since it extends to the third case, it will have two imaginary roots. Then, because of $B = 2$, $C = 8$, $D = 9$, it will be

$$\sqrt{CC - \frac{32}{9}BD} = \sqrt{64 - 64} = 0.$$

Therefore, compare $D = 9$ to $\frac{3}{16} \cdot 8\sqrt[3]{-8} = -3$. Therefore, because $D = 9 > -3$, also the two remaining roots will be imaginary.

## EXAMPLE 5

*Let this equation be propounded* $x^4 - 4x^3 - 7x^2 + 34x - 24 = 0$, *whose roots are known to be* 1, 2, 4 *and* $-3$.

But if we apply the rules, having removed the second term by putting $x = y + 1$, it will be

$$y^4 - 13yy + 12y + 0 = 0,$$

which compared to the second case gives $B = 13$, $C = 12$ and $D = 0$. Therefore, it has to be $D > \frac{3}{16} \cdot 24\sqrt[3]{-24}$ or $0 > -9\sqrt[3]{3}$ and $D < 0$; therefore, because $D$ is not larger than 0, the equation is indicated to have four real roots. For, if $D = 0$, the other equation will go over into

$$D < \frac{3}{16}\left(\frac{16BD}{9C}\right)\sqrt[3]{4C} \quad \text{and hence} \quad 1 < \frac{B}{3C}\sqrt[3]{4C}$$

or $27CC < 4B^3$; but $27 \cdot 144 < 4 \cdot 13^3$ or $36 \cdot 27 > 13^3$.

**§310** It would be very difficult, to transfer these things to equations of higher degree, since the roots of the differential equations cannot be exhibited in most cases; but if it is possible to assign these roots, from the given principles it is easily concluded, how many real and imaginary roots the propounded equation has. Hence the roots of all equations, which consist only of three terms, can be determined, whether they are real or imaginary. For, let this general equation be propounded

$$x^{m+n} + Ax^n + B = 0 = z.$$

Take its differential

$$\frac{dz}{dx} = (m+n)x^{m+n-1} + nAx^{n-1}; \tag{1}$$

Having put it equal to zero, it will at first be $x^{n-1} = 0$; hence, if $n$ was an odd number, a root exhibiting a maximum or a minimum results; but if $n$ is an even number, the root $x = 0$ is to be taken into account. But then it will be $(m+n)x^m + nA = 0$; if $m$ is an even number and $A$ a positive quantity, this equation has no real root. Hence the following cases are to be considered.

I. Let $m$ be an even number and $n$ an odd number and the root $x = 0$ will not exist. Therefore, if $A$ was a positive quantity, one will have no root exhibiting a maximum or a minimum; hence, because of the odd number $m+n$, the propounded equation will have one single real root. But if $A$ was a negative quantity, say $A = -E$, it will be $x = \pm \sqrt[m]{\frac{nE}{m+n}}$, whence

$$\alpha = +\sqrt[m]{\frac{nE}{m+n}} \quad \text{and} \quad \beta = -\sqrt[m]{\frac{nE}{m+n}}.$$

From these values

$$\mathfrak{A} = (x^m - E)x^n + B = -\frac{nE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$$

and

$$\mathfrak{B} = +\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{m:n} + B.$$

Therefore, if $\mathfrak{A}$ was a negative quantity or

$$\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} > B,$$

the equation will have one single real root $> \alpha$. If additionally

$$B > -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m},$$

i.e., by combining both conditions into a single one, if it was

$$(m+n)^{m+n}B^m < m^m n^n E^{m+n},$$

the equation will have three real roots, and if this condition is not satisfied, only one root of the equation will be real. These conditions are satisfied for the equation $x^{m+n} - Ex^n + B = 0$, if $m$ was an even number and $n$ an odd number; if $E$ was a negative number here, the equation will always have one single real root.

II. Let both numbers $m$ and $n$ be odd that $m+n$ is an even number and the root $x = 0$ is not to be taken into account. Since $(m+n)x^m + nA = 0$, it will be $x = -\sqrt[m]{\frac{nA}{m+n}}$; if one root is $= \alpha$, it will be

$$\mathfrak{A} = \frac{mA}{m+n}x^n + B = -\frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B.$$

If this value was negative, the propounded equation will have two real roots, otherwise none. Therefore, the propounded equation $x^{m+n} + Ax^n + B = 0$ will have two real roots, if it was

$$m^m n^n A^{m+n} > (m+n)^{m+n} B^m;$$

if it was

$$m^m n^n A^{m+n} < (m+n)^{m+n} B^m,$$

no root will be real.

III. Let the two numbers $m$ and $n$ be even; likewise, $m+n$ will be an even number and the root $x = 0$ will yield a maximum or a minimum; it will be the only one, if $A$ was a positive quantity, whence, having put $\alpha = 0$, it will be $\mathfrak{A} = B$. Hence, if $B$ also was a positive quantity, the equation will have no real root; but if $B$ is also a negative quantity, one will have two real roots and not more, if $A$ was a positive quantity. But let us put that $A$ is a negative quantity or $A = -E$; it will be $x = \pm\sqrt[m]{\frac{nE}{m+n}}$ and we will have three maxima or minima, namely

$$\alpha = \pm\sqrt[m]{\frac{nE}{m+n}}, \quad \beta = 0, \quad \gamma = -\sqrt[m]{\frac{nE}{m+n}}.$$

The following values correspond to these values of $z = x^{m+n} - Ex^n + B$

$$\mathfrak{A} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B, \quad \mathfrak{B} = B, \quad \mathfrak{C} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B.$$

Therefore, if $B$ is a negative quantity, because of the negative $\mathfrak{A}$ and $\mathfrak{C}$, the equation will have only two real roots, since also $\mathfrak{B} = B$ becomes negative. But if $B$ was a positive quantity, the equation will have four real roots, if

$$(m+n)^{m+n} B^m < m^m n^n E^{m+n}.$$

But it will have no real root, if it was

$$(m+n)^{m+n} B^m > m^n n^n E^{m+n}.$$

IV. Let $m$ be an odd number and $n$ an even number and the root $x = 0$ will give a maximum or a minimum. Furthermore, it will be $x = -\sqrt[m]{\frac{nA}{m+n}}$. Therefore, if $A$ is a positive number, it will be $\alpha = 0$ and $\beta = -\sqrt[m]{\frac{nA}{m+n}}$ and hence

$$\mathfrak{A} = B \quad \text{and} \quad \mathfrak{B} = \frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B.$$

Hence, if $B$ is a negative quantity, say $B = -F$, and additionally

$$m^m n^n A^{m+n} > (m+n)^{m+n} F^m,$$

the equation will have three real roots; otherwise only one will be real. But if $A$ is a negative quantity, say $A = -E$, it will be $x = +\sqrt[m]{\frac{nE}{m+n}}$ and

$$\alpha = \sqrt[m]{\frac{nE}{m+n}} \quad \text{and} \quad \beta = 0,$$

to which these correspond

$$\mathfrak{A} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B \quad \text{and} \quad \mathfrak{B} = B.$$

Hence the equation will have three real roots, if $B$ was a positive quantity and

$$m^m n^n E^{m+n} > (m+n)^{m+n} B^m;$$

if this property is not present, the propounded equation will have one single root.

**§311** Let all coefficients be $= 1$ and while $\mu$ and $\nu$ denote integer numbers the decision for the following equations will be made this way:

$$x^{2\mu+2\nu-1} + x^{2\nu-1} \pm 1 = 0$$

will have one single real root.

$$x^{2\mu+2\nu-1} - x^{2\nu-1} \pm 1 = 0$$

will have three real roots, if it was

$$(2\mu + 2\nu - 1)^{2\mu+2\nu-1} < (2\mu)^{2\mu}(2\nu - 1)^{2\nu-1};$$

since this can never happen, the equation will always have one single real root.

$$x^{2\mu+2\nu} \pm x^{2\nu-1} - 1 = 0$$

has two real roots.

$$x^{2\mu+2\nu} \pm x^{2\nu-1} + 1 = 0$$

has no real root.

$$x^{2\mu+2\nu} \pm x^{2\nu} + 1 = 0$$

has no real root.

$$x^{2\mu+2\nu} + x^{2\nu} \pm 1 = 0$$

has one single real root.

$$x^{2\mu+2\nu} - x^{2\nu} \pm 1 = 0$$

has one single real root.

Furthermore, since in the third case both exponents are even, putting $xx = y$ it can be reduced to a simpler form and hence this case could have been omitted. Having done this, one will be able to affirm that no equation consisting of three terms can have more than three real roots.

## EXAMPLE

*Let the cases be in question in which this equation* $x^5 \pm Ax^2 \pm B = 0$ *has three real roots.*

Since this equation extends to the fourth case, it is plain that the quantities $A$ and $B$ must have opposite signs. Hence, if it does not have a form of this kind, it will have only one real root; but if the propounded equation was of this kind $x^5 \pm Ax^2 \mp B = 0$, for it to have three real roots, it is necessary that $3^3 2^2 A^5 > 5^5 B^3$ or $A^5 \frac{3125}{108} B^3$. Therefore, if it was $B = 1$, it is necessary that $A^5 > \frac{3125}{108}$ or $A > 1.960132$. Therefore, if $A = 2$, this equation $x^5 - 2x^2 + 1 = 0$ has three real roots; since one of them is $x = 1$, it follows that this fourth order equation $x^4 + x^3 + x^2 - x - 1 = 0$ has two real roots. This can be understood both from the given prescriptions and from the results demonstrated in the first part of the book; for, there we showed that any equation if even degree whose absolute term is negative always has two real roots.

**§312** Using these principles one can also consider equations which consist of four terms, if the roots of the differential equations can be exhibited in a convenient way, what happens, if the exponents of $x$ in the three consecutive terms following are terms in an arithmetic progression. But since this consideration in general leads to several cases, let us treat them in some examples.

## EXAMPLE 1

*Let this equation be propounded* $x^7 - 2x^5 + x^3 - a = 0$.

Having put $z = x^7 - 2x^5 + x^3 - a$, it will be

$$\frac{dz}{dx} = 7x^6 - 10x^4 + 3xx,$$

having set which value equal to zero it will at first be $xx = 0$, which double value is to be treated as none. But then it will be $7x^4 = 10x^3 - 3$, whence $x^2 = \frac{5 \pm 2}{7}$, and four values for $x$ will emerge, which, ordered according to their magnitude, will yield the following values for $z$:

$$\begin{array}{lll|lll} \alpha & = & 1 & \mathfrak{A} & = & -a \\ \beta & = & +\sqrt{\dfrac{3}{7}} & \mathfrak{B} & = & \dfrac{48}{343}\sqrt{\dfrac{3}{7}} - a \\ \gamma & = & -\sqrt{\dfrac{3}{7}} & \mathfrak{C} & = & \dfrac{-48}{343}\sqrt{\dfrac{3}{7}} - a \\ \delta & = & -1 & \mathfrak{D} & = & -a. \end{array}$$

Therefore, if $a$ is a positive number, it will be either $a > \frac{48}{343}\sqrt{\frac{3}{7}}$ or $a < \frac{48}{343}\sqrt{\frac{3}{7}}$; in the first case, because of the all negative quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, the propounded equation will have one single root $x > 1$. In the second case, if $a < \frac{48}{343}\sqrt{\frac{3}{7}}$, the equation will have three real roots, the first $> 1$, the second contained within the limits 1 and $\sqrt{\frac{3}{7}}$ and the third within the limits $+\sqrt{\frac{3}{7}}$ and $-\sqrt{\frac{3}{7}}$.

But if $a$ is a negative quantity, putting $x = -y$, the equation will be reduced to the first form. Therefore, for the propounded equation to have three real roots it is necessary that $a < 0.0916134$ or $a < \frac{1}{11}$.

## EXAMPLE 2

*Let this equation be propounded $ax^8 - 3x^6 + 10x^3 - 12 = 0$.*

Since here the exponents of the three last terms are terms of an arithmetic progression, put $x = \frac{1}{y}$ and the equation will be transformed into this one

$$a - 3y^2 + 10y^5 - 12y^8 = 0;$$

therefore, put

$$z = 12y^8 - 10y^5 + 3y^2 - a = 0$$

and by differentiation it will be

$$\frac{dz}{dy} = 96y^7 - 50y^4 + 6y = 0,$$

from which equation at first $y = 0$; then, it will be

$$y^6 = \frac{50y^3 - 6}{96} \quad \text{and} \quad y^3 = \frac{25 \pm 7}{96}$$

and hence either $y = \sqrt[3]{\frac{1}{3}}$ or $y = \sqrt[3]{\frac{3}{16}}$. Therefore, having ordered these values according to their magnitude, the corresponding values of $z$ will be as follows:

$$\begin{array}{lcl|lcl}
\alpha & = & \sqrt[3]{\dfrac{1}{3}} & \mathfrak{A} & = & \sqrt[3]{\dfrac{1}{9}} - a \\
\beta & = & \sqrt[3]{\dfrac{3}{16}} & \mathfrak{B} & = & \dfrac{99}{64}\sqrt[3]{\dfrac{9}{256}} - a = \dfrac{99}{256}\sqrt[3]{\dfrac{9}{4}} - 1 \\
\gamma & = & 0 & \mathfrak{C} & = & -a.
\end{array}$$

Therefore, if it was $a > \sqrt[3]{\frac{1}{9}}$, the propounded equation will have two real roots, the one $> \sqrt[3]{\frac{1}{3}}$, the other $< 0$; but except for these it will additionally have two real roots, if at the same time $\mathfrak{B}$ was a positive quantity, i.e., if it was $a < \frac{99}{256}\sqrt[3]{\frac{9}{4}}$. Therefore, the propounded equation will have four real roots, if the quantity $a$ is contained within the limits $\sqrt[3]{\frac{1}{9}}$ and $\frac{99}{256}\sqrt[3]{\frac{9}{4}}$; these limits are approximately 0.48075 and 0.50674. Therefore, having put $a = \frac{1}{2}$, this equation $x^8 - 6x^6 + 20x^3 - 24 = 0$ will have four real roots within the limits $\infty$, $\sqrt[3]{\frac{3}{16}}$, $\sqrt[3]{3}$, $0$, $-\infty$; therefore, three will be positive and one negative.